Background of the Author

The author, Eric Holcomb, has an MS degree in aerospace engineering, and worked as an engineer for The Boeing Company for nearly 20 years. Eric now pursues his own business activities, enjoys playing chess, and serves as business manager for Northwest Chess magazine.

So How Many Chess Board Positions Are There?
By Eric Holcomb
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Perhaps some of you may recall hearing or reading at one time or another that there are about 10^{40} board positions and 10^{120} move sequences in the game of chess, where for example 10^{40} is the number "1" followed by 40 zeroes – a very large number indeed! (10^{120} is very much larger still – greater than the number of atoms in the known universe!)

In game theory, the complexity of a board game is often expressed in terms of the "state space complexity" (number of legal board positions reachable from the initial position), and the "game-tree size" (number of move sequences). Chess is indeed a very complex game, although not the most complex. Among major board games, that distinction belongs to the Japanese game of Go played on the traditional 19x19 board. (See the Wikipedia article on "Game Complexity.")

In this, the first of two articles, the author will discuss the number of chess board positions, and various estimates of this quantity. In a second article, the author will report on a remarkable coincidence involving one estimate of the number of chess board positions, and also discuss the number of chess move sequences.

Although it's been known for many years that the number of possible chess board positions is at least 10^{40}, an exact calculation of the number of legal positions is impossible in practice because of complications like pawn structure and both kings in check. The best that can be done is to calculate "upper bounds" based on various assumptions (for example, neglecting pawn structure, checks, pawn promotions to new pieces, etc.) The resulting estimates can vary widely depending on the assumptions!

The theory of combinatorial statistics underlies all of the calculations. For example, start with an empty board with 48 squares accessible to pawns, and place four identical white pawns on the board. The number of distinct arrangements (statistical combinations, not chess combinations!) is 48 x 47 x 46 x 45 / (1 x 2 x 3 x 4) = 194,580. Continuing with four black pawns gives 44 x 43 x 42 x 41 / (1 x 2 x 3 x 4) = 135,751 combinations for each white pawn arrangement, for a total (actually a product) of 194,580 x 135,751 = 26,414,429,580. The possibilities are already in the billions!

Statisticians would normally write the numbers given above in "factorial" notation, for example 48! / (44! x 4!) = 194,580, where 48! (48 factorial) is the product of all positive integers from 1 to 48, etc. The factor of 4! in the denominator arises because all white pawns (or all black pawns) are assumed to be identical and interchangeable. (Never mind that one of them came from a different chess set; that doesn't count!)

In 1950, computer chess pioneer Claude Shannon (1916-2001) used this statistical theory to calculate the number of chess board positions with all 32 chessmen on the board, without regard to the rules of chess. (In other words, the chessmen may be placed on any squares, as long as each man occupies a different square.) The resulting number is written as $64! / 32!(8!)^2(2!)^6$, which has a numerical value of 4.635×10^{42}, or 9.270×10^{42} (almost 10^{43}) if the result is doubled because chess positions are considered distinct depending on whose turn it is to move.

The actual number of chessmen on the board will be less than 32 once one or more captures have taken place. There is no single formula to handle this complexity in the calculation of board positions. Instead a computer program with "nested" loops is required to sum up all the possibilities. Won't this program take an impossibly long time to run? No. – The computer is not looping on the actual board positions, but rather on how many of each piece type are on the board, possibly with some extra work to account for light and dark squared bishops, pawn promotion, or other considerations. For each loop, the statistical formula is applied and the results summed.

The author developed a Visual Basic for Excel program (macro) that executes 10,497,600 such loops in a few seconds on a modern PC. The author's assumptions were as follows:

(1) All pieces (except kings) may be either on or off the board.
(2) Pawns may be placed anywhere on the 2nd thru 7th ranks without regard to the moves (e.g., captures) that would be required to get them there.
(3) Bishops must remain on their original (light or dark) color squares.
(4) Obtaining new pieces (e.g., more than one queen) by pawn promotion is not allowed. Chess legend Philidor approved of this rule!!
(5) Positions are counted without regard to one or both kings in check, including checking each other.
(6) Each position counts twice due to whose turn it is to move (again, without regard to checks).
(7) Loss of the ability to castle or capture "en passant" is not considered.

In statistical theory, the same answer should result no matter what order the pieces are placed on the board, but it makes sense to place the pawns first, then the bishops, because these pieces require extra bookkeeping. (Pawns are restricted to 48 squares, and bishops require accounting separately for light and dark squares. Once the bishops are placed, no further accounting of light and dark squares is necessary for the other pieces.) Readers with programming experience may be able to understand how the author's macro performs the required calculations. An Excel file with the macro will be available on the Northwest Chess website, nwchess.com, when this article is published in the magazine.

The answer is approximately 4.1529×10^{40} board positions. 14 decimal places are calculated in Excel, but some will be meaningless due to round-off errors. With commercially available mathematics software that performs unlimited precision arithmetic, it should be possible to calculate all 41 digits of the answer exactly! (Please let the author know if you try it.)

Of the 4.1529×10^{40} calculated board positions, 1.1057×10^{40}, or about one-quarter, involve all 32 chessmen on the board. This is over 800 times smaller than Shannon's estimate, in part because of the restriction on bishops, but mostly because pawns cannot be on the 1st or 8th ranks. (Think about it or try it – place 16 pawns on the board at random, and less than 1% of the time will any two ranks (or files) specified in advance be completely empty of pawns.)

It's true that the vast majority of these board positions are simply impossible in a real game played from the standard initial position, or even a "Chess 960" position. For example, of the 1.1057×10^{40} positions that involve all 32 men on the board, the author estimates that only 1 in 10 million (about 10^{33} positions) have "no capture" pawn structures with one black and one white pawn on each file. (For each file, there are only 15 possible arrangements of the white and black pawn before any captures have taken place. For all 8 files, that gives $(15)^8 = 2.563 \times 10^9$ arrangements of the pawns.) But this limitation can be removed by playing the original version of "bughouse" chess where captured chessmen can be put back almost anywhere on the board. (Not that crazy two-board version of "bughouse" chess that kids play today!)

Many more chess board positions are possible by allowing promotion to multiple queens or other pieces. The Wikipedia article on the "Shannon number" (10^{120} move sequences) quotes recent (1994) estimates by Victor Allis in a Ph.D. thesis of 5×10^{52} for an "upper bound" on the number of chess positions, and about 10^{50} for the "true number" of legal positions. Wow! That's a lot more than the 10^{40} or so positions without pawn promotion!

To check this out, the author modified his Excel macro to include pawn promotions to other pieces, while removing the restriction on bishops. The results, summarized in the table below, are quite interesting. (Note the use of the "E" notation for powers of 10 commonly used in computer input/output.)

Est.	Assumptions	# Loops	# Board Positions
1	Pawns on 2nd to 7th ranks, bishops on original color squares, no pawn promotion to new pieces; 2 to 32 men; positions count twice due to turn to move.	10,497,600	4.1529E+40
2	Same as #1 above except bishops unrestricted. There are far fewer loops because it is no longer necessary to keep track of light and dark squares.	236,196	1.4445E+41
3	Shannon estimate (for comparison); all 32 chessmen on board; no pawn promotion; no other restrictions.	n/a	9.2695E+42
4	Same as #2 above except pawn promotion to one (and only one) of the following piece types allowed: rooks, knights, bishops.	1,285,956	5.3227E+44
5	Same as #2 above except pawn promotion to queens allowed.	2,125,764	4.6250E+45
6	Same as #2 above except pawn promotion to all four piece types allowed.	75,585,636	4.7875E+49
7	Same as #6 above except pawns may be placed on any rank, even 1st or 8th ranks without promotion.	75,585,636	1.8983E+50

The author's highest estimate of 1.8983×10^{50} positions (just under 10^{50} without the double counting) is consistent with the Allis estimate of 10^{50} positions, but not having read Allis' Ph.D. thesis, the author is not certain how Allis derived his "upper bound" of 5×10^{52}, which seems unrealistically high. One thing, however, is clear: the vast majority of the “state space” of about 10^{50} chess board positions could only arise in practice through the cooperation of both players to bring about an absurd series of pawn promotions, resulting in lots of extra pieces on the board! Furthermore, the captures required to “clear the way” for pawn promotion would severely limit the number of possible positions. For example, of the 4.7875×10^{49} positions in line #6 above, 75% require that all of the original 32 chessmen (or their promoted equivalents) still be on the chessboard. However, as noted earlier, there are only about 10^{33} possible positions in a standard game of chess before any captures have taken place. Requiring that at least four black pawns must be captured to “clear the way” for all eight white pawns to promote reduces the number of possible board positions to about 10^{45}. (The flexibility of the author’s computer program makes it relatively easy to do those kinds of calculations.)

This is still a long way from answering the question of how many chess board positions can arise from a legal sequence of moves from the starting position. The author’s guess is that it’s closer to 10^{45} than 10^{50}, but the question is certainly worthy of further study. For players not satisfied with this immense level of complexity, there are always chess variants that increase the size of the “state space,” for example two-board bughouse, Capablanca chess, and Seirawan chess!

What about that remarkable coincidence mentioned? It involves the number 4.1529×10^{40} as discussed above. The coincidence does not involve the much larger number of atoms in the universe, but it does involve something similar of physical significance here on Earth. The author will award a prize of a one-year WCF or OCF membership to the first person who correctly identifies the coincidence before the author’s next article is published. (See the inside front cover of the magazine for contact information.)